

Non-singular modified gravity unifying inflation with late-time acceleration and universality of viscous ratio bound in $F(R)$ theory

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The review of basic cosmological properties of four-dimensional $F(R)$ -gravity, including FRW equations of motion and its accelerating solutions, generalized fluid and scalar-tensor representation of the theory is done. Cosmological reconstruction equation is written and conditions for stability of cosmological solution are discussed. The overview of realistic $F(R)$ -models unifying inflation with dark energy epoch is made. The avoidance of finite-time future singularities in such theories via the introduction of R^2 -term is studied. New realistic non-singular $F(R)$ -gravity unifying early-time inflation with late-time acceleration is presented. The exit from inflationary era in such model may be caused by the gravitational scenario. It is demonstrated that five-dimensional $F(R)$ -gravity considered as non-perturbative stringy effective action leads to universal relation for viscous bound ratio.

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I. INTRODUCTION

It is widely accepted that the universe evolution passed at least two accelerating epochs: the early-time inflation and current late-time acceleration. The striking similarity between these two accelerating eras indicates that their origin is caused by the same fundamental scenario. The most natural and elegant explanation of the known universe evolution in unified manner may be given within modified gravity approach. It does not request any extra dark components like inflaton, dark energy and dark matter which should be implemented into the theory. The role of these dark components is played by the gravitational theory which changes its functional form at different curvature scales. For instance, at the early universe the gravitational theory contains higher derivatives terms which cause the early-time inflation. During radiation/matter dominance higher derivatives terms are negligible if compare with standard General Relativity. Meanwhile, current dark energy epoch is due to the gravitational terms which dominate at very low curvature. The corresponding review on modified gravities unifying the early-time inflation and late-time acceleration is given in ref. [1]. The easiest version of modified gravity which naturally unifies the early-time inflation and late-time acceleration is $F(R)$ -theory [1, 2]. The number of versions of $F(R)$ -gravity is known to be consistent with astrophysical data and local tests (for review, see [3]).

In the present paper we study different models of $F(R)$ -gravity which may unify the early-time inflation with late-time acceleration in the consistent way. Moreover, we present new non-singular models for such unification. The paper is organized as follows. In the next section the general properties of $F(R)$ -gravity are reviewed. Spatially-flat FRW equations of motion are presented. These FRW equations are rewritten in the form of standard FRW equations for General Relativity with generalized $F(R)$ -fluid. Simple accelerating cosmologies are briefly discussed. Section III is devoted to the review of scalar-tensor representation of $F(R)$ -gravity (so-called Einstein frame). It is shown how to bound the (non-physical) anti-gravity regime in scalar-tensor representation of the theory. In section IV we propose general strategy for cosmological reconstruction of $F(R)$ -gravity, i.e. derivation of its functional form which contains given cosmological solution. Section V is devoted to the study of stability of given cosmological solution. It gives the cosmological solution conditions which define its stability/instability. In section VI we discuss viable $F(R)$ -gravities unifying inflation with dark energy. The conditions for derivation of such models are given. The review of the known realistic unified $F(R)$ -theories is made. It is shown that some of these models may contain the finite-time future singularity. Nevertheless, the addition of R^2 -term which supports the early-time inflation cures the future singularities in accord with the original proposal of refs. [4–6]. New non-singular viable $F(R)$ -gravity unifying the inflation with dark energy is proposed and investigated. It is shown that such theory has the instable inflationary era. The decay of the inflationary era is caused by the structure of gravitational theory. Section VII is devoted to the study of different, five-dimensional application of $F(R)$ -gravity. Here, we discuss five-dimensional $F(R)$ -gravity as some non-perturbative stringy gravity in frames of AdS/CFT correspondence. It is proved that for such theory like

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for five-dimensional General Relativity the viscous ratio bound is universal. Some summary and outlook are given in last section.

II. GENERAL PROPERTIES OF $F(R)$ -GRAVITY

Let us briefly review general properties of four-dimensional $F(R)$ -gravity which is known to be realistic candidate for the unification of early-time inflation with late-time acceleration (for review, see [1]). The starting action is chosen to be:

$$S_{F(R)} = \int d^4x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right). \quad (1)$$

For general $F(R)$ -gravity, one can define an effective equation of state (EoS) parameter. The spatially-flat FRW equations in Einstein gravity coupled with perfect fluid are:

$$\rho = \frac{3}{\kappa^2} H^2, \quad p = -\frac{1}{\kappa^2} (3H^2 + 2\dot{H}). \quad (2)$$

For modified gravity, one may define an effective EoS parameter as follows:

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}. \quad (3)$$

The equation of motion for modified $F(R)$ -gravity is given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \square F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2} T_{\text{matter } \mu\nu}. \quad (4)$$

Assuming spatially-flat FRW universe,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (5)$$

the FRW-like equations are given by

$$\rho_{\text{eff}} = \frac{3}{\kappa^2} H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa^2} (3H^2 + 2\dot{H}), \quad (6)$$

$$\rho_{\text{eff}} = \frac{1}{\kappa^2} \left(-\frac{1}{2} f(R) + 3(H^2 + \dot{H}) f'(R) - 18(4H^2 \dot{H} + H \ddot{H}) f''(R) \right) + \rho_{\text{matter}}, \quad (7)$$

$$p_{\text{eff}} = \frac{1}{\kappa^2} \left(\frac{1}{2} f(R) - (3H^2 + \dot{H}) f'(R) + 6(8H^2 \dot{H} + 4\dot{H}^2 + 6H \ddot{H} + \ddot{H}) f''(R) + 36(4H \dot{H} + \ddot{H})^2 f'''(R) \right) + p_{\text{matter}}. \quad (8)$$

There may be found several (often exact) solutions of (6). Nevertheless, due to presence of higher derivatives and non-linear terms, the structure of exact solutions is much more complicated than in General Relativity. Without any matter, assuming that the Ricci tensor is covariantly constant, that is, $R_{\mu\nu} \propto g_{\mu\nu}$, Eq. (4) reduces to the algebraic equation [7]:

$$0 = 2F(R) - RF'(R). \quad (9)$$

If Eq. (9) has a solution, the Schwarzschild (or Kerr) - (anti-)de Sitter or de Sitter space is an exact vacuum solution.

When $F(R)$ behaves as $F(R) \propto R^m$ and there is no matter, there appears the following solution:

$$H \sim \frac{-\frac{(m-1)(2m-1)}{m-2}}{t}, \quad (10)$$

which gives the following effective EoS parameter:

$$w_{\text{eff}} = -\frac{6m^2 - 7m - 1}{3(m-1)(2m-1)}. \quad (11)$$

When $F(R) \propto R^m$ again but if the matter with a constant EoS parameter w is included, one may get the following solution:

$$H \sim \frac{\frac{2m}{3(w+1)}}{t}, \quad (12)$$

and the effective EoS parameter is given by

$$w_{\text{eff}} = -1 + \frac{w+1}{m}. \quad (13)$$

This shows that modified gravity may describe early/late-time universe acceleration which could reproduce the quintessence-like or phantom-like behaviour. Realistic non-linear $F(R)$ models are discussed below.

III. SCALAR-TENSOR DESCRIPTION OF $F(R)$ -GRAVITY

It is well-known that one can rewrite $F(R)$ -gravity action in the scalar-tensor form. Introducing the auxiliary field A , we rewrite the action (1) of the $F(R)$ -gravity as:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \{F'(A) (R - A) + F(A)\}. \quad (14)$$

By the variation over A , one obtains $A = R$. Substituting $A = R$ into the action (14), one can reproduce the action (1). Furthermore, we rescale the metric in the following way (conformal transformation):

$$g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}, \quad \sigma = -\ln F'(A). \quad (15)$$

Hence, the Einstein frame action is obtained:

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right),$$

$$V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}. \quad (16)$$

Here $g(e^{-\sigma})$ is given by solving the equation $\sigma = -\ln(1 + f'(A)) = -\ln F'(A)$ as $A = g(e^{-\sigma})$. Due to the scale transformation (15), there appears a coupling of the scalar field σ with usual matter when matter is included. The mass of σ is given by

$$m_\sigma^2 \equiv \frac{3}{2} \frac{d^2 V(\sigma)}{d\sigma^2} = \frac{3}{2} \left\{ \frac{A}{F'(A)} - \frac{4F(A)}{(F'(A))^2} + \frac{1}{F''(A)} \right\}. \quad (17)$$

Unless m_σ is very large, there appears the large correction to the Newton law. Naively, one expects the order of the mass m_σ could be that of the Hubble rate, that is, $m_\sigma \sim H \sim 10^{-33}$ eV, which is very light and the correction could be very large.

In ref. [8] (see also ref. [9]), a “realistic” $F(R)$ model has been proposed. It may be shown however, that the model has a instability where the large curvature can be easily produced. This is the manifestation of the finite-time future singularity. In the model [8], a parameter $m \sim 10^{-33}$ eV with mass dimension is included. The parameter m plays a role of the effective cosmological constant. When the curvature R is large enough compared with m^2 , $R \gg m^2$, $F(R)$ [8] has the following approximate form:

$$F(R) = R - c_1 m^2 + \frac{c_2 m^{2n+2}}{R^n} + \mathcal{O}(R^{-2n}). \quad (18)$$

Here c_1 , c_2 , and n are positive dimensionless constants. Then the potential $V(\sigma)$ (16) has the following asymptotic form:

$$V(\sigma) \sim \frac{c_1 m^2}{A^2}. \quad (19)$$

Hence, the infinite curvature $R = A \rightarrow \infty$ corresponds to small value of the potential and therefore the large curvature can be easily produced.

Let us assume that when R is large, $F(R)$ behaves as

$$F(R) \sim F_0 R^\epsilon. \quad (20)$$

Here F_0 and ϵ are positive constant. One also assumes $\epsilon > 1$ so that this term dominates if compared with General Relativity. Then the potential $V(\sigma)$ (16) behaves as

$$V(\sigma) \sim \frac{\epsilon - 1}{\epsilon^2 F_0 R^{\epsilon-2}}. \quad (21)$$

Therefore if $1 < \epsilon < 2$, the potential $V(\sigma)$ diverges when $R \rightarrow \infty$ and therefore the large curvature is not realized so easily. When $\epsilon = 2$, $V(\sigma)$ takes finite value $1/F_0$ when $R \rightarrow \infty$. As long as $1/F_0$ is large enough, the large curvature could be prevented.

Note that there appears the anti-gravity regime when $F'(R)$ is negative, which follows from Eq. (14) of ref. [2] where first FR model unifying inflation with dark energy was proposed. Hence, one should require

$$F'(R) > 0. \quad (22)$$

As long as the condition (22) is satisfied there occurs usual gravity regime although there might appear extra force when m_σ^2 (17) is small. Note also that

$$\frac{dV(\sigma)}{dA} = \frac{F''(A)}{F'(A)^3} (-AF'(A) + 2F(A)). \quad (23)$$

Therefore if

$$0 = -AF'(A) + 2F(A), \quad (24)$$

the scalar field σ is on the local maximum or local minimum of the potential and therefore σ can be a constant. The condition (24) is nothing but the condition (9) for the existence of the de Sitter solution [7]. When the condition (24) is satisfied, the mass (17) can be rewritten as

$$m_\sigma^2 = \frac{3}{2F'(A)} \left(-A + \frac{F'(A)}{F''(A)} \right). \quad (25)$$

Then when the condition (22) for the exclusion of the anti-gravity is satisfied, the mass squared m_σ^2 is positive and therefore the scalar field is on the local minimum if

$$-A + \frac{F'(A)}{F''(A)} > 0. \quad (26)$$

On the other hand, if

$$-A + \frac{F'(A)}{F''(A)} < 0, \quad (27)$$

the scalar field is on the local maximum of the potential and the mass squared m_σ^2 is negative. As we will see later in (39), the condition (26) is nothing but the condition for the stability of the de Sitter space. Thus, scalar-tensor representation (Einstein frame) for $F(R)$ -gravity is constructed. Despite the mathematical equivalence of Jordan and Einstein frames pictures they lead to the theories which are not physically equivalent (for corresponding discussion, see [10]). (Or, more exactly, if physics in one frame is matched with observations and describe the observable accelerating universe then another frame physics becomes extremely unconventional. For instance, instead of acceleration the observer sees the deceleration, or matter couples with dilaton, etc.) Note also that it is evident how to transform $F(R)$ -gravity to the effective General Relativity with generalized $F(R)$ fluid [11]. Of course, the generalized fluid is gravitational fluid and it contains higher derivatives of curvature invariants. This fact is often ignored in the study of cosmological perturbations in $F(R)$ -gravity, which leads to basically erroneous conclusions or not well-justified approximations where fourth-order differential equations become second order ones similar to standard General Relativity cosmological perturbations (see, for instance, ref. [12]). For correct treatment the covariant, higher-derivative cosmological perturbations theory of $F(R)$ -gravity was developed in ref. [13]. However, it turns out that corresponding higher-derivative equations are much more complicated than the corresponding ones in General Relativity.

IV. COSMOLOGICAL RECONSTRUCTION OF MODIFIED $F(R)$ -GRAVITY

Let us demonstrate that any FRW cosmology may be realized in specific $F(R)$ -gravity. In other words, we propose the general solution for inverse problem, i.e. the cosmological reconstruction of $F(R)$ -gravity (for the introduction and review, see [14]). Let us rewrite the FRW equation (6) by using a new variable (which is often called e-folding) instead of the cosmological time t , $N = \ln \frac{a}{a_0}$. The variable N is related with the redshift z by

$$e^{-N} = \frac{a_0}{a} = 1 + z. \quad (28)$$

Since $\frac{d}{dt} = H \frac{d}{dN}$ and therefore $\frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$, one can rewrite (6) as (for details, see [15])

$$0 = -\frac{F(R)}{2} + 3(H^2 + HH')F'(R) - 18\left(4H^3H' + H^2(H')^2 + H^3H''\right)F''(R) + \kappa^2\rho_{\text{matter}}. \quad (29)$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$. If the matter energy density ρ_{matter} is given by a sum of the fluid densities with constant EoS parameter w_i , one finds

$$\rho_{\text{matter}} = \sum_i \rho_{i0} a^{-3(1+w_i)} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N}. \quad (30)$$

Let the Hubble rate is given in terms of N via the function $g(N)$ as

$$H = g(N) = g(-\ln(1+z)). \quad (31)$$

Then scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved with respect to N as $N = N(R)$. Using (30) and (31), one can rewrite (29) as

$$0 = -18\left(4g(N(R))^3 g'(N(R)) + g(N(R))^2 g'(N(R))^2 + g(N(R))^3 g''(N(R))\right) \frac{d^2 F(R)}{dR^2} + 3\left(g(N(R))^2 + g'(N(R))g(N(R))\right) \frac{dF(R)}{dR} - \frac{F(R)}{2} + \kappa^2 \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N(R)}, \quad (32)$$

which constitutes a differential equation for $F(R)$, where the variable is scalar curvature R . Instead of g , if we use $G(N) \equiv g(N)^2 = H^2$, the expression (32) could be a little bit simplified:

$$0 = -9G(N(R))(4G'(N(R)) + G''(N(R))) \frac{d^2 F(R)}{dR^2} + \left(3G(N(R)) + \frac{3}{2}G'(N(R))\right) \frac{dF(R)}{dR} - \frac{F(R)}{2} + \kappa^2 \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N(R)}. \quad (33)$$

Note that the scalar curvature is given by $R = 3G'(N) + 12G(N)$. Hence, when we find $F(R)$ satisfying the differential equation (32) or (33), such $F(R)$ theory admits the solution (31). The reconstructed $F(R)$ -gravity realizes any given above cosmological solution. For instance, one can present the forms of $F(R)$ -theory which describe inflation or dark energy or even unify the early-time inflation with late-time acceleration era. The number of corresponding examples may be explicitly realized. However, as a rule the corresponding theory is some complicated non-analytical function expressed in terms of special functions. Hence, the cosmological reconstruction turns out to be approximate one. The exact reconstruction of Λ CDM universe from pure $F(R)$ -gravity often leads to (almost) General Relativity [16].

V. STABILITY OF THE COSMOLOGICAL SOLUTION

In the previous section, we presented a formulation to obtain the form of $F(R)$ to reproduce the known solution $H = H(N)$. Note, however, the solution $H = H(N)$ is not always stable. In this section, the condition of the stability of such solution is studied. The FRW equation (6) can be rewritten by using $G = H^2$ and the e-folding N , as in (33):

$$0 = -9G(N)(4G'(N) + G''(N)) \left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'(N)+12G(N)} + \left(3G(N) + \frac{3}{2}G'(N)\right) \left. \frac{dF(R)}{dR} \right|_{R=3G'(N)+12G(N)} - \left. \frac{F(R)}{2} \right|_{R=3G'(N)+12G(N)} + \kappa^2 \rho_{\text{matter}}(N). \quad (34)$$

Let us assume the N -dependence of ρ_{matter} is known as in (30). Let a solution of (34) be $G = G_0(N)$ and consider the perturbation from the solution:

$$G(N) = G_0(N) + \delta G(N). \quad (35)$$

Then Eq. (34) gives

$$\begin{aligned} 0 = & G_0(N) \left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} \delta G''(N) + \left\{ 3G_0(N) (4G'_0(N) + G''_0(N)) \left. \frac{d^3 F(R)}{dR^3} \right|_{R=3G'_0(N)+12G_0(N)} \right. \\ & + \left. \left(3G_0(N) - \frac{1}{2}G'_0(N) \right) \left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} \right\} \delta G'(N) \\ & + \left\{ 12G_0(N) (4G'_0(N) + G''_0(N)) \left. \frac{d^3 F(R)}{dR^3} \right|_{R=3G'_0(N)+12G_0(N)} \right. \\ & + \left. (-4G_0(N) + 2G'_0(N) + G''_0(N)) \left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} + \frac{1}{3} \left. \frac{dF(R)}{dR} \right|_{R=3G'_0(N)+12G_0(N)} \right\} \delta G. \quad (36) \end{aligned}$$

Note that in this formulation, as it is assumed that the N -dependence of ρ_{matter} is known, one need not consider the fluctuation of ρ_{matter} . If the cosmological time t is used instead of N , we need to include the fluctuation of ρ_{matter} . Thus, the stability conditions are given by

$$6 (4G'_0(N) + G''_0(N)) \left. \frac{d^3 F(R)}{dR^3} \right|_{R=3G'_0(N)+12G_0(N)} \left(\left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} \right)^{-1} + 6 - \frac{G'_0(N)}{G_0(N)} > 0, \quad (37)$$

$$\begin{aligned} & 36 (4G'_0(N) + G''_0(N)) \left. \frac{d^3 F(R)}{dR^3} \right|_{R=3G'_0(N)+12G_0(N)} \left(\left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} \right)^{-1} - 12 + \frac{6G'_0(N)}{G_0(N)} \\ & + \frac{3G''_0(N)}{G_0(N)} + \frac{1}{G_0(N)} \left. \frac{dF(R)}{dR} \right|_{R=3G'_0(N)+12G_0(N)} \left(\left. \frac{d^2 F(R)}{dR^2} \right|_{R=3G'_0(N)+12G_0(N)} \right)^{-1} > 0. \quad (38) \end{aligned}$$

In case of de Sitter space, where H and therefore G_0 and $R = R_0 \equiv 12G_0$ are constants, Eq. (37) becomes $6 > 0$ and trivially satisfied and Eq. (38) becomes

$$-12G_0 + \left. \frac{dF(R)}{dR} \right|_{R=12G_0} \left(\left. \frac{d^2 F(R)}{dR^2} \right|_{R=12G_0} \right)^{-1} = -R_0 + \left. \frac{dF(R)}{dR} \right|_{R=R_0} \left(\left. \frac{d^2 F(R)}{dR^2} \right|_{R=R_0} \right)^{-1} > 0, \quad (39)$$

which is the standard result.

One may consider the case that a form of $F(R)$ admits two de Sitter solutions. If one is stable but another is unstable, there could be a solution that matches the unstable de Sitter solution to the stable solution. If the Hubble rate H of the unstable solution is much larger than the Hubble rate of the stable one, the unstable solution may correspond to the inflation and the stable one to the late-time acceleration. Alternatively, we may directly construct $F(R)$ -gravity which admits the transition from asymptotic de Sitter universe with the large Hubble rate to another asymptotic de Sitter universe with the small Hubble rate. Then if the corresponding solution of $F(R)$ -theory satisfies the conditions (37) and (38), such a transition surely occurs. One should bear in mind that other scenarios for exit from inflationary phase (for instance, due to quantum effects) may be proposed as well.

Let us now consider the stability of de Sitter universe satisfying the condition (9), or equivalently (24). Eq. (9) can be rewritten as

$$0 = \frac{d}{dR} \left(\frac{F(R)}{R^2} \right). \quad (40)$$

Let $R = R_0$ is a solution of (40). Then $F(R)$ has the following form:

$$\frac{F(R)}{R^2} = f_0 + \tilde{f}(R) (R - R_0)^n. \quad (41)$$

Here f_0 is a constant, which should be positive if $F(R) > 0$ and n is an integer greater or equal to two: $n \geq 2$. We assume the function $\tilde{f}(R)$ does not vanish at $R = R_0$, $\tilde{f}(R_0) \neq 0$. When $n = 2$, it follows

$$-R_0 + \frac{F'(R_0)}{F''(R_0)} = -\frac{\tilde{f}(R_0)A_0}{f_0 + f(A_0)}. \quad (42)$$

Eq. (39) shows that the de Sitter solution is stable if

$$-f_0 < f(A_0) < 0. \quad (43)$$

When $n \geq 3$, one gets

$$-R_0 + \frac{F'(R_0)}{F''(R_0)} = 0. \quad (44)$$

Hence, we need more detailed investigation to check the stability. One now uses the expression of m_σ^2 in (17) and investigates the sign of m_σ in the region $R \sim R_0$. The expression (25) cannot be used since it is only valid in the point $R = R_0$. Therefore

$$m_\sigma^2 \sim -\frac{3n(n-1)R_0^2\tilde{f}(R_0)}{2f_0^2}(R-R_0)^{n-2}. \quad (45)$$

Eq. (45) shows that when n is an even integer, the de Sitter solution is stable if $\tilde{f}(R_0) < 0$ but unstable if $\tilde{f}(R_0) > 0$. On the other hand, when n is an odd integer, the de Sitter solution is always unstable. Note, however, that when $\tilde{f}(R_0) < 0$ ($\tilde{f}(R_0) > 0$), we find $m_\sigma^2 > 0$ ($m_\sigma^2 < 0$) if $R > R_0$ but $m_\sigma^2 < 0$ ($m_\sigma^2 > 0$) if $R < R_0$. Therefore when $\tilde{f}(R_0) < 0$, R becomes smaller but when $\tilde{f}(R_0) > 0$, R becomes larger.

Thus, it is developed the scheme which permits to check when the unification of inflation with dark energy is realistic one because inflationary stage is instable.

VI. VIABLE $F(R)$ -GRAVITIES UNIFYING INFLATION WITH DARK ENERGY

A. The known realistic $F(R)$ -models unifying inflation with dark energy

In refs. [17], [18], and [19], viable models of $F(R)$ -gravity unifying the late-time acceleration and the inflation were proposed. To construct these models, we have required several conditions:

1. In order to generate the inflation, one may require

$$\lim_{R \rightarrow \infty} f(R) = -\Lambda_i. \quad (46)$$

Here Λ_i is an effective cosmological constant at the early universe and therefore it is natural to assume $\Lambda_i \gg (10^{-33}\text{eV})^2$. For instance, it is natural to have $\Lambda_i \sim 10^{20 \sim 38} (\text{eV})^2$.

2. In order that the current cosmic acceleration could be generated, the current $f(R)$ -gravity is considered to be a small constant, that is,

$$f(R_0) = -2\tilde{R}_0, \quad f'(R_0) \sim 0. \quad (47)$$

Here R_0 is the current curvature $R_0 \sim (10^{-33}\text{eV})^2$. Note that $R_0 > \tilde{R}_0$ due to the contribution from matter. In fact, if we can regard $f(R_0)$ as an effective cosmological constant, the effective Einstein equation gives $R_0 = \tilde{R}_0 - \kappa^2 T_{\text{matter}}$. Here T_{matter} is the trace of the matter energy-momentum tensor. Note that $f'(R_0)$ need not to vanish exactly. Since the time scale of one-ten billion years is considered, we only require $|f'(R_0)| \ll (10^{-33}\text{eV})^4$. Instead of the model corresponding to (46), one may consider a model which satisfies

$$\lim_{R \rightarrow \infty} f(R) = \alpha R^m, \quad (48)$$

with a positive integer $m > 1$ and a constant α . In order to avoid the anti-gravity $f'(R) > -1$, we find $\alpha > 0$ and therefore $f(R)$ should be positive at the early universe. On the other hand, Eq. (47) shows that $f(R)$ is negative at the present universe. Therefore $f(R)$ should cross zero in the past.

3. The last condition is

$$\lim_{R \rightarrow 0} f(R) = 0, \quad (49)$$

which means that there is a flat space-time solution.

One typical model proposed in ref. [17] and satisfying the above conditions is

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1 + \gamma R^n}. \quad (50)$$

Here α , β , and γ are positive constants and n is a positive integer. Eq. (50) gives [17]

$$R_0 = \left\{ \left(\frac{1}{\gamma} \right) \left(1 + \sqrt{1 + \frac{\beta\gamma}{\alpha}} \right) \right\}^{1/n}, \quad (51)$$

and therefore

$$f(R_0) \sim -2\tilde{R}_0 = \frac{\alpha}{\gamma^2} \left(1 + \frac{\left(1 - \frac{\beta\gamma}{\alpha} \right) \sqrt{1 + \frac{\beta\gamma}{\alpha}}}{2 + \sqrt{1 + \frac{\beta\gamma}{\alpha}}} \right). \quad (52)$$

Then it follows

$$\alpha \sim 2\tilde{R}_0 R_0^{-2n}, \quad \beta \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{n-1}, \quad \gamma \sim 2\tilde{R}_0 R_0^{-2n} R_I^{n-1}. \quad (53)$$

In the model (50), the correction to the Newton law could be small since the mass m_σ (17) is large and is given by $m_\sigma^2 \sim 10^{-160+109n} \text{ eV}^2$ in the solar system and $m_\sigma^2 \sim 10^{-144+98n} \text{ eV}^2$ in the air on the earth [17]. In both cases, the mass m_σ is very large if $n \geq 2$.

As a model corresponding to (46), we proposed the theory [19]

$$f(R) = -\alpha_0 \left(\tanh \left(\frac{b_0(R - R_0)}{2} \right) + \tanh \left(\frac{b_0 R_0}{2} \right) \right) - \alpha_I \left(\tanh \left(\frac{b_I(R - R_I)}{2} \right) + \tanh \left(\frac{b_I R_I}{2} \right) \right).$$

One now assumes

$$R_I \gg R_0, \quad \alpha_I \gg \alpha_0, \quad b_I \ll b_0, \quad (54)$$

and

$$b_I R_I \gg 1. \quad (55)$$

When $R \rightarrow 0$ or $R \ll R_0$, R_I , $f(R)$ behaves as

$$f(R) \rightarrow - \left(\frac{\alpha_0 b_0}{2 \cosh^2 \left(\frac{b_0 R_0}{2} \right)} + \frac{\alpha_I b_I}{2 \cosh^2 \left(\frac{b_I R_I}{2} \right)} \right) R. \quad (56)$$

and $f(0) = 0$ again. When $R \gg R_I$, it follows

$$f(R) \rightarrow -2\Lambda_I \equiv -\alpha_0 \left(1 + \tanh \left(\frac{b_0 R_0}{2} \right) \right) - \alpha_I \left(1 + \tanh \left(\frac{b_I R_I}{2} \right) \right) \sim -\alpha_I \left(1 + \tanh \left(\frac{b_I R_I}{2} \right) \right). \quad (57)$$

On the other hand, when $R_0 \ll R \ll R_I$, one gets

$$f(R) \rightarrow -\alpha_0 \left[1 + \tanh \left(\frac{b_0 R_0}{2} \right) \right] - \frac{\alpha_I b_I R}{2 \cosh^2 \left(\frac{b_I R_I}{2} \right)} \sim -2\Lambda_0 \equiv -\alpha_0 \left[1 + \tanh \left(\frac{b_0 R_0}{2} \right) \right]. \quad (58)$$

Here, we have assumed the condition (55). One also finds

$$f'(R) = -\frac{\alpha_0 b_0}{2 \cosh^2 \left(\frac{b_0(R - R_0)}{2} \right)} - \frac{\alpha_I b_I}{2 \cosh^2 \left(\frac{b_I(R - R_I)}{2} \right)}, \quad (59)$$

which has two valleys when $R \sim R_0$ or $R \sim R_I$. When $R = R_0$,

$$f'(R_0) = -\alpha_0 b_0 - \frac{\alpha_I b_I}{2 \cosh^2 \left(\frac{b_I(R_0 - R_I)}{2} \right)} > -\alpha_I b_I - \alpha_0 b_0. \quad (60)$$

On the other hand, when $R = R_I$, it follows

$$f'(R_I) = -\alpha_I b_I - \frac{\alpha_0 b_0}{2 \cosh^2 \left(\frac{b_0(R_0 - R_I)}{2} \right)} > -\alpha_I b_I - \alpha_0 b_0. \quad (61)$$

Due to the condition (22) to avoid the anti-gravity period, one obtains

$$\alpha_I b_I + \alpha_0 b_0 < 2. \quad (62)$$

In the solar system domain, on or inside the earth, where $R \gg R_0$, $f(R)$ can be approximated by

$$f(R) \sim -2\Lambda_{\text{eff}} + 2\alpha e^{-b(R-R_0)}. \quad (63)$$

On the other hand, since $R_0 \ll R \ll R_I$, by assuming Eq. (55), $f(R)$ (54) could be also approximated by

$$f(R) \sim -2\Lambda_0 + 2\alpha e^{-b_0(R-R_0)}, \quad (64)$$

which has the same expression, after having identified $\Lambda_0 = \Lambda_{\text{eff}}$ and $b_0 = b$. Hence, one may check the case (63) only. The effective mass has the following form

$$m_\sigma^2 \sim \frac{e^{b(R-R_0)}}{4\alpha b^2}, \quad (65)$$

which could be very large, that is, $m_\sigma^2 \sim 10^{1,000} \text{ eV}^2$ in the solar system and $m_\sigma^2 \sim 10^{10,000,000,000} \text{ eV}^2$ in the air surrounding the earth, and the correction to the Newton law becomes negligible.

One may consider another model [17]:

$$f(R) = -\frac{(R - R_0)^{2k+1} + R_0^{2k+1}}{f_0 + f_1 \left\{ (R - R_0)^{2k+1} + R_0^{2k+1} \right\}}. \quad (66)$$

It has been shown [17] that for $k \geq 10$ such modified gravity passes the local tests. It also unifies the early-time inflation with dark energy epoch. In (66), R_0 is current curvature $R_0 \sim (10^{-33} \text{ eV})^2$. We also require

$$f_0 \sim \frac{R_0^{2n}}{2}, \quad f_1 = \frac{1}{\Lambda_i}. \quad (67)$$

Here Λ_i is the effective cosmological constant in the inflation epoch. When $R \gg \Lambda_i$, $f(R)$ (66) behaves as

$$f(R) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 R^{2n+1}}. \quad (68)$$

The trace equation, which is the trace part of (4) looks as:

$$3\Box f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T_{\text{matter}}. \quad (69)$$

Here T_{matter} is the trace of the matter energy-momentum tensor. For FRW metric with flat spatial part (5), one finds

$$R \sim (t_0 - t)^{-2/(2n+3)}, \quad (70)$$

which diverges at finite future time $t = t_0$. By a similar analysis, we can show that if $f(R)$ behaves as $f(R) \sim R^\epsilon$ for large R with a constant ϵ , a future singularity appears if $\epsilon > 2$ or $\epsilon < 0$, which is consistent with the analysis (21) via the scalar-tensor form of the action. Conversely if $2 \geq \epsilon \geq 0$, the singularity does not appear. Hence, adding the term $R^2 \tilde{f}(R)$, where $\lim_{R \rightarrow 0} \tilde{f}(R) = c_1$, $\lim_{R \rightarrow \infty} \tilde{f}(R) = c_2$, to $f(R)$ (66), the future singularity (70) disappears.

Let us observe the above situation in more detail. Now we assume

$$f(R) \sim F_0 + F_1 R^\epsilon, \quad (71)$$

when R is large. Here F_0 and F_1 are constants where F_0 may vanish but $F_1 \neq 0$. In case of (68)

$$F_0 = -\frac{1}{f_1}, \quad F_1 = \frac{f_0}{f_1^2}, \quad \epsilon = -(2n+1). \quad (72)$$

Under the assumption (71), the trace equation (69) gives

$$3F_1 \square R^{\epsilon-1} = \begin{cases} R & \text{when } \epsilon < 0 \text{ or } \epsilon = 2 \\ (2-\epsilon) F_1 R^\epsilon & \text{when } \epsilon > 1 \text{ or } \epsilon \neq 2 \end{cases}. \quad (73)$$

In the FRW background with flat spatial part (5), when the Hubble rate has a singularity as

$$H \sim \frac{h_0}{(t_0 - t)^\beta}, \quad (74)$$

with constants h_0 and β , the scalar curvature $R = 6\dot{H} + 12H^2$ behaves as

$$R \sim \begin{cases} \frac{12h_0^2}{(t_0 - t)^{2\beta}} & \text{when } \beta > 1 \\ \frac{6h_0 + 12h_0^2}{(t_0 - t)^2} & \text{when } \beta = 1 \\ \frac{6\beta h_0}{(t_0 - t)^{\beta+1}} & \text{when } \beta < 1 \end{cases}. \quad (75)$$

In (74) or (75), $\beta \geq 1$ case corresponds to Type I (Big Rip) singularity in [20], $1 > \beta > 0$ to Type III, $0 > \beta > -1$ to Type II, and $\beta < -1$ but $\beta \neq \text{integer}$ to Type IV.

The classification of finite-time future singularities used above is given in ref. [20]:

- Type I (“Big Rip”) : For $t \rightarrow t_s$, $a \rightarrow \infty$, $\rho_{\text{eff}} \rightarrow \infty$ and $|p_{\text{eff}}| \rightarrow \infty$. This also includes the case of ρ_{eff} , p_{eff} being finite at t_s .
- Type II (“sudden”) : For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$ and $|p_{\text{eff}}| \rightarrow \infty$
- Type III : For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \infty$ and $|p_{\text{eff}}| \rightarrow \infty$
- Type IV : For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow 0$, $|p_{\text{eff}}| \rightarrow 0$ and higher derivatives of H diverge. This also includes the case in which p_{eff} (ρ_{eff}) or both of p_{eff} and ρ_{eff} tend to some finite values, while higher derivatives of H diverge.

Here ρ_{eff} and p_{eff} are defined by

$$\rho_{\text{eff}} \equiv \frac{3}{\kappa^2} H^2, \quad p_{\text{eff}} \equiv -\frac{1}{\kappa^2} (2\dot{H} + 3H^2). \quad (76)$$

By substituting (75) into (73), one finds that there are two classes of consistent solutions. The first solution is specified by $\beta = 1$ and $\epsilon > 1$ but $\epsilon \neq 2$ case, which corresponds to the Big Rip ($h_0 > 0$ and $t < t_0$) or Big Bang ($h_0 < 0$ and $t > t_0$) singularity at $t = t_0$. Another one is $\epsilon < 1$, and $\beta = -\epsilon/(\epsilon - 2)$ ($-1 < \beta < 1$) case, which corresponds to (70) and to the II Type future singularity. In fact, we find $\epsilon = -2n - 1$ and therefore $\beta + 1 = -2/(2n + 3)$. We should note that when $\epsilon = 2$, that is, $f(R) \sim R^2$, there is no any singular solution. Therefore if we add the above term $R^2 \tilde{f}(R)$, where $\lim_{R \rightarrow 0} \tilde{f}(R) = c_1$, $\lim_{R \rightarrow \infty} \tilde{f}(R) = c_2$, to $f(R)$ in (66), the added term dominates and modified $f(R)$ behaves as $f(R) \sim R^2$, the future singularity (70) disappears. We also note that if we add R^n -term with $n = 3, 4, 5, \dots$, the singularity becomes (in some sense) worse since this case corresponds to $\epsilon = n > 1$, that is Big Rip case. Using the potential which appears when we transform $F(R)$ -gravity to scalar-tensor theory [5], it has been found that the future singularity may not appear in case $0 < \epsilon < 2$.

Thus, in order to avoid the finite-time future singularity one has to add R^2 -term to above viable unification models:

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1 + \gamma R^n} + cR^2, \quad (77)$$

$$f(R) = -\alpha_0 \left(\tanh \left(\frac{b_0(R - R_0)}{2} \right) + \tanh \left(\frac{b_0 R_0}{2} \right) \right) - \alpha_I \left(\tanh \left(\frac{b_I(R - R_I)}{2} \right) + \tanh \left(\frac{b_I R_I}{2} \right) \right) + cR^2, \quad (78)$$

$$f(R) = -\frac{(R - R_0)^{2k+1} + R_0^{2k+1}}{f_0 + f_1 \left\{ (R - R_0)^{2k+1} + R_0^{2k+1} \right\}} + cR^2. \quad (79)$$

The addition of this term has been proposed first in ref. [4] where it was shown that in this case the Big Rip singularity disappears. Moreover, such term which effectively cures future singularity also supports the early-time inflation. In other words, adding such R^2 -term to gravitational dark energy model may lead to emergence of inflationary phase in the model as it was first observed in ref. [2]. In case, when model already contains the inflationary era, its dynamics will be changed by R^2 -term. The investigation which shows that it cures all types of future singularity has been done in refs. [5, 6]. In fact, it was realized lately that some phenomenological problems [21] of $F(R)$ -dark energy (like consistent description of neutron stars) may be resolved in the presence of R^2 -term. Moreover, the traditional phantom/quintessence (fluid/scalar) dark energy models often bring the universe to finite-time future singularity. It has been demonstrated in ref. [22] that the natural prescription to cure singularity in this models is again the addition of R^2 -term. In other words, curing future singularity requests from specific dark energy model to be (at least, partly) the modified gravity!

B. New viable $F(R)$ -models

The above analysis shows that, in order to obtain a realistic and viable model, $F(R)$ -gravity should satisfy the following conditions:

1. When $R \rightarrow 0$, the Einstein gravity is recovered, that is,

$$F(R) \rightarrow R \quad \text{that is,} \quad \frac{F(R)}{R^2} \rightarrow \frac{1}{R}. \quad (80)$$

This also means that there is a flat space solution as in (49).

2. As discussed after Eq. (45), there appears stable de Sitter solution, which corresponds to the late-time acceleration and therefore the curvature is small $R \sim R_L \sim (10^{-33} \text{ eV})^2$. This requires, when $R \sim R_L$,

$$\frac{F(R)}{R^2} = f_{0L} - f_{1L} (R - R_L)^{2n+2} + o\left((R - R_L)^{2n+2}\right). \quad (81)$$

Here f_{0L} and f_{1L} are positive constants and n is a positive integer.

3. As also discussed after Eq. (45), there appears quasi-stable de Sitter solution, which corresponds to the inflation in the early universe and therefore the curvature is large $R \sim R_I \sim (10^{16 \sim 19} \text{ GeV})^2$. The de Sitter space should not be exactly stable so that the curvature decreases very slowly. This require

$$\frac{F(R)}{R^2} = f_{0I} - f_{1I} (R - R_I)^{2m+1} + o\left((R - R_I)^{2m+1}\right). \quad (82)$$

Here f_{0I} and f_{1I} are positive constants and m is a positive integer.

4. Following the discussion after (20), when $R \rightarrow \infty$, in order to avoid the curvature singularity, it is proposed

$$F(R) \rightarrow f_\infty R^2 \quad \text{that is} \quad \frac{F(R)}{R^2} \rightarrow f_\infty. \quad (83)$$

Here f_∞ is a positive and sufficiently small constant. Instead of (83), we may take

$$F(R) \rightarrow f_\infty R^{2-\epsilon} \quad \text{that is} \quad \frac{F(R)}{R^2} \rightarrow \frac{f_\infty}{R^\epsilon}. \quad (84)$$

Here f_∞ is a positive constant and $0 < \epsilon < 1$. The above condition (83) or (84) prevents both of the future singularity and the singularity at high density matter.

5. As in (22), in order to avoid the anti-gravity, we require

$$F'(R) > 0, \quad (85)$$

which is rewritten as

$$\frac{d}{dR} \left(\ln \left(\frac{F(R)}{R^2} \right) \right) > -\frac{2}{R}. \quad (86)$$

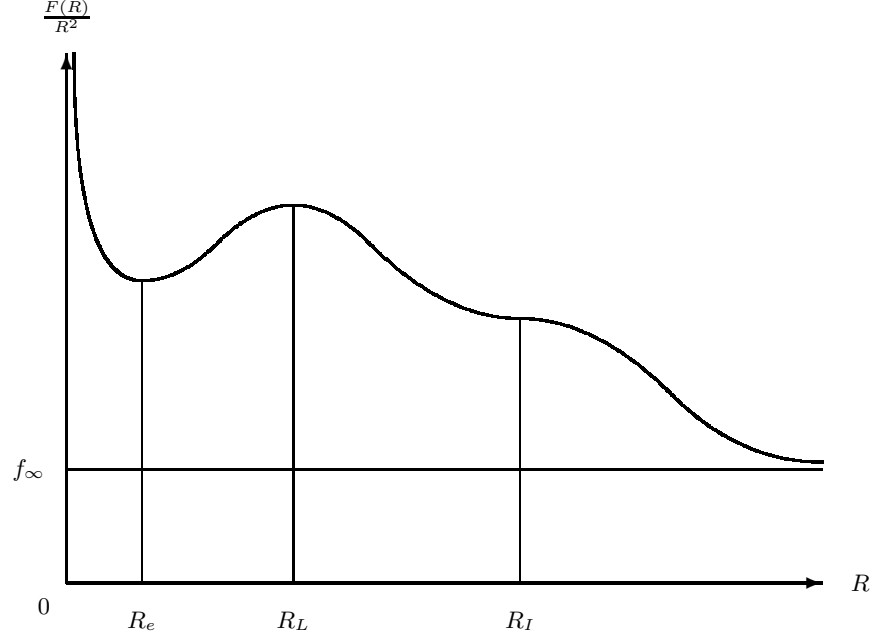


FIG. 1: The qualitative behavior of $\frac{F(R)}{R^2}$ versus R for a viable model.

6. Combining the conditions (80) and (85), one finds

$$F(R) > 0. \quad (87)$$

The conditions 1 and 2 show that there must appear unstable extra de Sitter solution at $R = R_e$ ($0 < R_e < R_L$), see Fig.1. Since the universe evolution will stop at $R = R_L$ since the de Sitter solution $R = R_L$ is stable, the curvature never becomes smaller than R_L and therefore the extra de Sitter solution is not realized. The behavior of $\frac{F(R)}{R^2}$ which satisfies the conditions (80), (81), (82), (83), and (87) is given in Fig.1.

An example of such $F(R)$ -gravity is

$$\begin{aligned} \frac{F(R)}{R^2} &= \left\{ (X_m(R_I; R) - X_m(R_I; R_1)) (X_m(R_I; R) - X_m(R_I; R_L))^{2n+2} \right. \\ &\quad \left. + X_m(R_I; R_1) X_m(R_I; R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{1}{2n+3}}, \\ X_m(R_I; R) &\equiv \frac{(2m+1) R_I^{2m}}{(R - R_I)^{2m+1} + R_I^{2m+1}}. \end{aligned} \quad (88)$$

Here n and m are integers greater or equal to unity, $n, m \geq 1$ and R_1 is a parameter related with R_e by

$$X(R_I; R_e) = \frac{(2n+2) X(R_I; R_1) X(R_I; R_1) + X(R_I; R_L)}{2n+3}, \quad (89)$$

It is assumed

$$0 < R_1 < R_L \ll R_I. \quad (90)$$

$X(R_I; R_e)$ is a monotonically decreasing function of R and in the limit $R \rightarrow 0$, $X(R_I; R_e)$ behaves as

$$X(R_I; R_e) \rightarrow \frac{1}{R}, \quad (91)$$

which tells that in the limit, $F(R)$ (88) is (80) and therefore the condition 1 is satisfied. On the other hand, when $R \rightarrow \infty$,

$$X(R_I; R_e) \rightarrow \frac{(2m+1) R_I^{2m}}{R^{2m+1}} \rightarrow 0, \quad (92)$$

and therefore $F(R)$ behaves as (83) and the condition 4 is satisfied.

When $R \sim R_L$, we find the behavior of (81), where

$$\begin{aligned} f_{0L} &= \left\{ X_m(R_I; R_1) X_m(R_I; R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{1}{2n+3}}, \\ f_{1L} &= \frac{1}{2m+3} \left\{ X_m(R_I; R_1) X_m(R_I; R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{-\frac{2(n+1)}{2n+3}} (X_m(R_I; R_1) - X_m(R_I; R_L)) \\ &\quad \times \frac{(2m+1)^{4(m+1)} \{R_I(R_L - R_I)\}^{4m(m+1)}}{\left\{ (R_L - R_I)^{2m+1} + R_I^{2m+1} \right\}^{4(m+1)}}. \end{aligned} \quad (93)$$

Then the condition 2 is satisfied.

On the other hand, when $R \sim R_I$, we also find the behavior of (82), where

$$\begin{aligned} f_{0I} &= \left\{ (X_m(R_I; R_I) - X_m(R_I; R_1)) (X_m(R_I; R_I) - X_m(R_I; R_L))^{2n+2} \right. \\ &\quad \left. + X_m(R_I; R_1) X_m(R_I; R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{1}{2n+3}} \\ f_{1I} &= \frac{2m+1}{R_I^{2m+2}} \left\{ (X_m(R_I; R_I) - X_m(R_I; R_1)) (X_m(R_I; R_I) - X_m(R_I; R_L))^{2n+2} \right. \\ &\quad \left. + X_m(R_I; R_1) X_m(R_I; R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{2(n+1)}{2n+3}} \left\{ (X_m(R_I; R_I) - X_m(R_I; R_L))^{2n+2} \right. \\ &\quad \left. + (2n+2) (X_m(R_I; R_I) - X_m(R_I; R_1)) (X_m(R_I; R_I) - X_m(R_I; R_L))^{2n+1} \right\}. \end{aligned} \quad (94)$$

Hence, the condition 3 is satisfied.

We now investigate the mass of the scalar field σ in order to check if the Chameleon mechanism [23] works or not. For this purpose, one investigates the region $R_1 < R_L \ll R \ll R_I$. In this region, $X_m(R_I; R)$ can be approximated as

$$X_m(R_I; R) \sim \frac{1}{R}, \quad X_m(R_I; R_1) \sim \frac{1}{R_1}, \quad X_m(R_I; R_L) \sim \frac{1}{R_L}. \quad (95)$$

Then $F(R)$ can be approximated as

$$\frac{F(R)}{R^2} \sim f_\infty + \frac{f_n}{R}, \quad f_n \equiv \frac{R_1 + (2n+2) R_L f_\infty^{-2n-2}}{(2n+3) R_1 R_L^{2n+2}} \sim \frac{f_\infty^{-2n-2}}{R_L^{2n+2}}. \quad (96)$$

In the last equation, it was assumed $R_1 \sim R_L$. If we assume $f_\infty R \ll f_n$ or

$$\frac{1}{f_\infty^{2n+3}} \gg R R_L^{2n+2}, \quad (97)$$

one gets

$$m_\sigma^2 \sim \frac{3}{4f_\infty}. \quad (98)$$

If we consider the region inside the earth, since $1g \sim 6 \times 10^{32} \text{ eV}$ and $1 \text{ cm} \sim (2 \times 10^{-5} \text{ eV})^{-1}$, the density is about $\rho \sim 1 \text{ g/cm}^3 \sim 5 \times 10^{18} \text{ eV}^4$. This shows that the magnitude of the curvature is $R \sim \kappa^2 \rho \sim (10^{-19} \text{ eV})^2$. In the air on the earth, one finds $\rho \sim 10^{-6} \text{ g/cm}^3 \sim 10^{12} \text{ eV}^4$, which gives $R_0 \sim \kappa^2 \rho \sim (10^{-25} \text{ eV})^2$. In the solar system, there could be interstellar gas. Typically, in the interstellar gas, there is one proton (or hydrogen atom) per 1 cm^3 , which shows $\rho \sim 10^{-5} \text{ eV}^4$, $R_0 \sim 10^{-61} \text{ eV}^2$. Then, the condition (97) can be easily satisfied, for example, we can chose

$\frac{1}{f_\infty} \sim \text{MeV}^2$. Then the Compton length of the scalar field becomes very small and the correction to the Newton law is negligible.

There remains the condition 5. Since $F(R)$ (88) satisfies Eqs. (80) and (83), it is clear that Eq. (85) or (86) and therefore the condition 5 are satisfied when $R \rightarrow 0$ or $R \rightarrow \infty$. Since $\frac{F(R)}{R^2}$ is a monotonically increasing function of R in the region $R_e < R < R_L$, Eq. (85) or (86) is trivially satisfied in the region. In the region $R_1 < R_L \ll R \ll R_I$, since $\frac{F(R)}{R^2}$ behaves as (96), we find Eq. (85) or (86) is satisfied again. Then the condition 5 seems to be satisfied in all the region.

After the inflation at $R = R_I$, the radiation and the matter are generated. If the energy densities of the radiation and the matter dominate compared with the contribution from $f(R) = R(R) - R$, that is, only the first term in (7) dominates, the radiation/matter dominated universe will be realized. The densities of the radiation and the matter decrease rapidly compared with the contribution from $f(R)$ -term in the late-time and when the curvature arrives at $R \sim R_L$, the late-time acceleration occurs.

The condition (84) indicates that there appears R^2 term in the very high curvature region. The R^2 -term will generate the inflation besides the inflation at $R = R_I$ if the universe started with very high curvature $R \gg R_I$. Since the inflation due to R^2 -term is unstable, the inflation at the very early stage would stop but when the curvature decreases and reaches $R \sim R_I$, there will occur the inflationary phase again.

Thus, we demonstrated that number of viable $F(R)$ -gravity models may explain the early-time inflation with dark energy epoch in unified way.

VII. UNIVERSALITY OF VISCOUS RATIO BOUND IN 5D $F(R)$ -GRAVITY

In the present section, we discuss the possible applications of 5d modified $F(R)$ -gravity as non-perturbative stringy gravity in AdS/CFT correspondence, which has provided a gravity dual framework to analyze strongly coupled gauge theory. Specifically, shear viscosity in the hydrodynamical limit of the gauge theories has been well studied [24–27]. It was found that the ratio of the coefficient of the shear viscosity η and the entropy density s should be larger than $1/4\pi$,

$$\frac{\eta}{s} \geq \frac{1}{4\pi}, \quad (99)$$

in the Einstein gravity. It turns out this bound is not universal. The correction to the bound coming from the higher order corrections, (Einstein-Gauss-Bonnet gravity or R^3 gravity,) has been calculated in refs. [28–30]. In this section, we consider $F(R)$ -gravity as a prototype non-linear model which may come from non-perturbative effects of string theory. We show the universality of the viscous bound ratio emerges even in this case. The non-universality of such bound is due to account of the effects in the next order of loop expansion, but our results indicate that for non-linear non-perturbative stringy model this may be not true.

Let us start from $F(R)$ -gravity in D dimensions:

$$I = \int d^D x \sqrt{-g} F(R). \quad (100)$$

Later one may choose $D = 5$. Here we do not consider the contribution from matter. Then from (4), the equation of the motion is given by

$$0 = \frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) + \nabla_\mu \nabla_\nu F'(R) - g_{\mu\nu} \nabla^2 F'(R). \quad (101)$$

If we assume the curvatures are covariantly constant

$$R = R_0, \quad R_{\mu\nu} = \frac{R_0}{D} g_{\mu\nu}, \quad (102)$$

with a constant R_0 , Eq. (101) reduces to an algebraic equation:

$$0 = \frac{D}{2} F(R_0) - R_0 F'(R_0), \quad (103)$$

which could be solved with respect to R_0 . This shows that any vacuum solution in the Einstein gravity like Schwarzschild solution, Kerr solution, gravitational wave, etc. is the solution of $F(R)$ -gravity. Especially if $R_0 < 0$,

we find Schwarzschild-anti-de Sitter (S-AdS)

$$ds_D^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 \sum_{i=1}^{D-2} (dx^i)^2, \quad V(r) = -\frac{M}{r^{D-3}} + \frac{r^2}{l^2}. \quad (104)$$

Kerr-anti-de Sitter (K-AdS), and Black string-anti de Sitter solutions. Here the length scale l is related with R_0 by

$$R_0 = -\frac{D(D-1)}{l^2}. \quad (105)$$

Eq. (104) shows that the horizon radius r_0 is given by

$$r_0 = (l^2 M)^{\frac{1}{D-1}}. \quad (106)$$

If the radial coordinate r is redefined as

$$\rho \equiv \frac{1}{r^2}, \quad (107)$$

the metric (104) is rewritten as

$$ds_D^2 = -\frac{1 - Ml^2\rho^{\frac{D-1}{2}}}{l^2\rho}dt^2 + \frac{l^2 d\rho^2}{4\rho^2 \left(1 - Ml^2\rho^{\frac{D-1}{2}}\right)} + \frac{1}{\rho}\Omega^2. \quad (108)$$

We now expand the action (100) from the solution $g_{\mu\nu} = g_{\mu\nu}^{(0)}$ (103)

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}. \quad (109)$$

Up to the third order in $h_{\mu\nu}$:

$$\begin{aligned} S = & \int d^D x \sqrt{-g} \left[F(R_0) - \frac{1}{4} F(R_0) h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} \left(-\frac{R_0 F'(R_0)}{2D} + \frac{R_0^2 F''(R_0)}{D^2} \right) h^2 \right. \\ & - \frac{1}{2} \left(F'(R_0) - \frac{R_0 F''(R_0)}{D} \right) h \nabla^{(0)\mu} \nabla^{(0)\mu} h_{\mu\nu} - \frac{1}{2} \left(-\frac{F'(R_0)}{2} + \frac{2R_0 F''(R_0)}{D} \right) h \left(\nabla^{(0)} \right)^2 h \\ & - \frac{F'(R_0)}{4} h^{\mu\nu} \left\{ \left(\nabla^{(0)} \right)^2 h_{\mu\nu} - \nabla^{(0)\rho} \left(\nabla_{\mu}^{(0)} h_{\nu\rho} + \nabla_{\nu}^{(0)} h_{\mu\rho} \right) \right\} \\ & \left. - \frac{1}{2} F''(R_0) \left\{ \nabla^{(0)\mu} \nabla^{(0)\mu} h_{\mu\nu} - \left(\nabla^{(0)} \right)^2 h \right\}^2 \right]. \quad (110) \end{aligned}$$

Here

$$h \equiv g^{(0)\mu\nu} h_{\mu\nu}, \quad h^{\mu\nu} \equiv g^{(0)\mu\rho} g^{(0)\nu\sigma} h_{\rho\sigma}, \quad (111)$$

and $\nabla_{\mu}^{(0)}$ expresses the covariant derivative given by $g_{\mu\nu}^{(0)}$. Then the viscosity coefficient η can be read from the coefficient in front of $(\partial_r h_{ij})$ which is contained only in the 6-th term as $-(F'(R_0)/4) h^{\mu\nu} \left(\nabla^{(0)} \right)^2 h_{\mu\nu}$

$$\eta = F'(R_0). \quad (112)$$

In case of $F(R)$ -gravity, the entropy is given by [7]

$$S = 4\pi A F'(R_0), \quad (113)$$

which gives the entropy density s as

$$s = \frac{S}{A} = 4\pi F'(R_0). \quad (114)$$

Combining (114) with (112), one gets

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad (115)$$

which is not changed from the case of the Einstein gravity. This result is completely consistent with the ones in [31] and [32].

As an example, we consider R^2 -gravity with a cosmological constant

$$F(R) = \frac{1}{2\kappa^2} \left\{ R + \frac{(D-1)(D-2)}{l_\Lambda^2} + cR^2 \right\}. \quad (116)$$

Here the cosmological constant Λ is given by

$$\Lambda = -\frac{(D-1)(D-2)}{l_\Lambda^2}. \quad (117)$$

Then (103) gives

$$R_0 = -\frac{D(D-1)}{l^2} = -\frac{D-2}{2c(D-4)} \left\{ 1 \pm \sqrt{1 - \frac{4cD(D-1)(D-4)}{(D-2)l_\Lambda^2}} \right\}. \quad (118)$$

In the signs \pm of (118), the minus sign $-$ reduces to the Einstein gravity case in $c \rightarrow 0$ limit. When the minus sign is chosen, in the limit we find

$$R_0 = -\frac{D(D-1)}{l_\Lambda^2} \left\{ 1 + \frac{cD(D-1)(D-4)}{(D-2)l_\Lambda^2} + \mathcal{O}(c^2) \right\}. \quad (119)$$

Then (112) gives

$$\eta = \frac{1}{2\kappa^2} (1 + 2cR_0) \sim \frac{1}{2\kappa^2} \left(1 - \frac{2cD(D-1)}{l_\Lambda^2} \right). \quad (120)$$

Especially when $D = 5$, one obtains

$$\eta \sim \frac{1}{2\kappa^2} \left(1 - \frac{40c}{l_\Lambda^2} \right). \quad (121)$$

This result essentially agrees with the previous ones in [27, 31, 32]. Note that in [32], the horizon radius was shifted from the Einstein gravity case ($c = 0$) by adding the correction term ($c \neq 0$), via account of the shift of the length parameter l_Λ in (117) corresponding to the Einstein gravity to l in (118), which includes the correction coming from c . The shift can be found by using (118) and (119).

$$l = l_\Lambda \left\{ 1 - \frac{cD(D-1)(D-4)}{2(D-2)l_\Lambda^2} + \mathcal{O}(c^2) \right\}, \quad (122)$$

which gives, in $D = 5$,

$$l = l_\Lambda \left\{ 1 - \frac{10c}{3l_\Lambda^2} + \mathcal{O}(c^2) \right\}. \quad (123)$$

When $D = 5$, Eq. (106) shows that r_0 is given by $r_0 = l^{\frac{1}{2}} M^{\frac{1}{4}}$. Then the horizon radius r_0 may be shifted as $r_0 \rightarrow (l/l_\Lambda)^{\frac{1}{2}} r_0$ by the correction from cR^2 term and therefore the area A_0 of the horizon as $A_0 \rightarrow (l/l_\Lambda)^{\frac{3}{2}} A_0$. Since the quantity η is the density on the horizon, we may regard η is also shifted as

$$\eta \rightarrow (l/l_\Lambda)^{\frac{3}{2}} \eta = \frac{1}{2\kappa^2} \left(1 - \frac{45c}{l_\Lambda^2} \right), \quad (124)$$

which completely agrees with the previous ones in refs. [27, 31, 32]. We should note, however, that since M in (104) is the constant of the integration the shift of the horizon radius can be always absorbed into the redefinition of M .

Therefore in this section, we may fix the value of the radius, which makes difference in the expression of the viscosity coefficient η but as we have seen, the ratio of η and the entropy density in (115) does not depend on the radius and exactly agrees with the results in [27, 31, 32]. This is because both of the viscosity coefficient η and the entropy density s are densities, they shift in an identical way by the shift of the horizon. For example, if we change the constant of the integration from M_1 to M_2 , η and s shift as

$$\eta \rightarrow (M_2/M_1)^{\frac{3}{4}} \eta, \quad s \rightarrow (M_2/M_1)^{\frac{3}{4}} s. \quad (125)$$

Then in the viscous ratio the shift is canceled with each other. Hence, we proved the emergence of the universality viscous ratio bound for 5d $F(R)$ -gravity. This may be understood also due to the fact of mathematical equivalence of $F(R)$ -gravity with scalar-tensor theory which also satisfies to universal viscous ratio bound. Nevertheless, other non-perturbative stringy gravity models (like $F(G)$ gravity [33]) should be investigated in order to understand if the restoration of universality of viscous ratio bound is the common property of non-perturbative models.

VIII. DISCUSSION

In summary, brief introduction to cosmological aspects of $F(R)$ -gravity is made. Spatially-flat FRW equations and its typical accelerating solutions are discussed. The generalized (gravitational) fluid as well as scalar-tensor description of the theory under investigation is given. The emergence of the Newton law in scalar-tensor formulation is described. Cosmological reconstruction technique for $F(R)$ -gravity is overviewed as well as general description of stability condition for cosmological solution. The review of known viable $F(R)$ -gravities unifying the inflation with dark energy epoch is given. Some of such models may lead to finite-time future singularities which may be cured by the addition of R^2 -term relevant at high curvature. It is remarkable that this term not only cures future singularity but also contributes to realization of inflationary era. In fact, the alternative gravity dark energy model which does not describe inflation may also lead to inflationary phase after addition of R^2 -term.

New realistic non-singular $F(R)$ -gravity unifying inflation with dark energy is introduced and studied in detail. It is shown that inflationary epoch is unstable and decays due to gravitational action structure. Hence, modified gravity provides not only the universal unification scenario but also the gravitational exit from inflation.

Section VI addresses different, five-dimensional application of $F(R)$ -gravity as non-perturbative stringy gravity within AdS/CFT correspondence. It is less related with previous sections which are devoted to cosmological study of the four-dimensional theory. It is demonstrated here that the well-known viscous ratio bound in five-dimensional $F(R)$ -gravity is universal unlike to the Einstein-Gauss-Bonnet gravity or stringy gravity with higher-order perturbative gravitational corrections.

Modified gravity unifying inflation with late-time acceleration has passed number of different checks, related with theoretical considerations, local tests or observational data. It is still considered as most natural candidate for description of the whole universe evolution in unified way. It is remarkable that just the same models may be used for inflation-dark energy unified description for totally different classes of theories. For instance, such unification turns out to be possible in recent Hořava-Lifshitz $F(R)$ -gravity where unified models were introduced in ref. [34].

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